THE ZEROTH L^2-HOMOLOGY OF COMPACT QUANTUM GROUPS

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Abstract. We prove that the zeroth L^2-Betti number of a compact quantum group vanishes unless the underlying C^*-algebra is finite dimensional, and that the zeroth L^2-homology itself is non-trivial exactly when the quantum group is coamenable.

1. Introduction and Notation

The results in this note is a continuation of the work carried out in [Kye08b] and [Kye08a] concerning L^2-homology and L^2-Betti numbers for compact quantum groups. Although the definitions will be given below, the reader not familiar with these concepts might benefit from casting a sidelong glance at [Kye08b] when reading the present text.

Consider a compact quantum group $\mathbb{G}$ in the sense of Woronowicz [Wor98]; i.e. $\mathbb{G}$ consists of a (not necessarily commutative) unital C^*-algebra $C(\mathbb{G})$ together with a unit-preserving $*$-homomorphism $\Delta_{\mathbb{G}}\colon C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ which furthermore has to be coassociative and satisfy a certain non-degeneracy condition. See e.g. [Wor98] for more details. We remind the reader that such a C^*-algebraic quantum group automatically gives rise to a purely algebraic quantum group (i.e. a Hopf $*$-algebra [KS97]) whose underlying algebra will be denoted $\mathrm{Pol}(\mathbb{G})$, as well as a von Neumann algebraic quantum group (see [KV03]) whose underlying algebra will be denoted $L^\infty(\mathbb{G})$. We also recall that the C^*-algebra $C(\mathbb{G})$ of a compact quantum group $\mathbb{G}$ comes with a distinguished state $h_{\mathbb{G}}$, called the Haar state, which plays the role corresponding to the Haar measure on a genuine, compact group. The compact quantum group is said to be of Kac type if its Haar state is a trace. Performing the GNS construction with respect to the Haar state yields a Hilbert space denoted $L^2(\mathbb{G})$ on which $C(\mathbb{G})$ acts via the corresponding GNS-representation λ.

Example 1.1. *The canonical example of a compact quantum group, on which the general definition is modeled, is obtained by considering a compact, Hausdorff topological group G and its C^*-algebra $C(G)$ of continuous, complex valued*

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functions. In this case the von Neumann algebra becomes $L^\infty(G,\mu)$ where μ denotes the Haar probability measure and the associated Hopf $*$-algebra becomes the algebra generated by matrix coefficients arising from the irreducible representations of G.

To any quantum group $\mathbb{G}$ (compact as well as non-compact) a so-called multiplicative unitary W on $L^2(\mathbb{G})\bar{\otimes}L^2(\mathbb{G})$ is associated; this is a unitary which (inter alia) has the property that

$$C(\mathbb{G})_{\mathrm{red}} \stackrel{\mathrm{def}}{=} \lambda(C(\mathbb{G})) = [(\mathrm{id}\otimes\omega)W \mid \omega \in B(L^2(\mathbb{G}))_*],$$

where, for a subset X of a normed space, $[X]$ denotes the norm closure of the linear space spanned by X. To any compact quantum group $\mathbb{G}$ a dual quantum group $\hat{\mathbb{G}}$ of so-called discrete type is associated. In its reduced form, this dual quantum group has as underlying C^*-algebra

$$C(\hat{\mathbb{G}})_{\mathrm{red}} = [(\omega\otimes\mathrm{id})W \mid \omega \in B(L^2(\mathbb{G}))_*], \tag{1}$$

and the associated multiplicative unitary is $\hat{W} = \Sigma W^*\Sigma$. For a more detailed treatment of C^*-algebraic (locally compact) quantum groups and their duality theory we refer to the work of Kustermans and Vaes in [KV00].

In [Kye08b] the notion of L^2-homology and L^2-Betti numbers was introduced in the setting of compact quantum groups of Kac type; if $\mathbb{G}$ is such a quantum group its L^2-homology and L^2-Betti numbers are defined as

$$H_n^{(2)}(\mathbb{G}) = \mathrm{Tor}_n^{\mathrm{Pol}(\mathbb{G})}(L^\infty(\mathbb{G}),\mathbb{C}) \quad \text{and} \quad \beta_n^{(2)}(\mathbb{G}) = \dim_{L^\infty(\mathbb{G})}(H_n^{(2)}(\mathbb{G})).$$

Here $\mathbb{C}$ is considered as a $\mathrm{Pol}(\mathbb{G})$-module via the counit $\varepsilon\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ and $\dim_{L^\infty(\mathbb{G})}(-)$ is Lück's extended Murray-von Neumann dimension (calculated with respect to the tracial Haar state) introduced in [Lüc98]. As shown in [Kye08b] Proposition 1.3, this extends the classical definition by means of the formula $\beta_n^{(2)}(\mathbb{G}) = \beta_n^{(2)}(\Gamma)$ when $C(\mathbb{G}) = C^*_{\mathrm{red}}(\Gamma)$ for a discrete group Γ.

The first aim of this note is to prove that zeroth L^2-Betti number of a compact quantum group vanishes unless the underlying C^*-algebra is finite dimensional; this is done in Section 2. In Section 3 the zeroth L^2-homology is examined and we prove that it vanishes exactly when $\mathbb{G}$ is non-coamenable.

Acknowledgements. Theorem 2.1 is an improvement of Proposition 2.2 in [Kye08b]. I thank Stefaan Vaes for pointing this improvement out to me.

Notation. Throughout the text, $\odot$ is used to denote algebraic tensor products which, unless specified otherwise, are assumed to be over the complex numbers. The symbol $\otimes$ is reserved to denote the minimal tensor product of C^*-algebras,

while $\bar{\otimes}$ will be used to denote the tensor product in the category of von Neumann algebras as well as the category of Hilbert spaces.

2. The zeroth L^2-Betti number

Cheeger and Gromov proved in [CG86] that if Γ is an infinite discrete group then $\beta_0^{(2)}(\Gamma) = 0$. We prove here the following quantum group analogue of this result, improving [Kye08b] Proposition 2.2.

Theorem 2.1. *If $\mathbb{G}$ is a compact quantum group of Kac type then $\beta_0^{(2)}(\mathbb{G}) = 0$ unless $\mathbb{G}$ is finite; i.e. unless $\dim_{\mathbb{C}}(C(\mathbb{G})) < \infty$.*

Proof. We assume $\mathbb{G}$ to be infinite and need to prove that $\beta_0^{(2)}(\mathbb{G}) = 0$. First note that

$$H_0^{(2)}(\mathbb{G}) = \operatorname{Tor}_0^{\operatorname{Pol}(\mathbb{G})}(L^\infty(\mathbb{G}), \mathbb{C}) \simeq L^\infty(\mathbb{G}) \underset{\operatorname{Pol}(\mathbb{G})}{\odot} \mathbb{C} \simeq L^\infty(\mathbb{G})/J,$$

where J is the left ideal in $L^\infty(\mathbb{G})$ generated by the kernel of the counit $\varepsilon\colon \operatorname{Pol}(\mathbb{G}) \to \mathbb{C}$. Denote by $\bar{J}$ the strong operator closure of J and note that

$$J \subseteq \bar{J} \subseteq \overline{J}^{\text{alg}} \overset{\text{def}}{=} \bigcap_{\substack{f \in L^\infty(\mathbb{G})^* \\ J \subseteq \ker(f)}} \ker(f),$$

where, for an $L^\infty(\mathbb{G})$-module X, we put $X^* = \operatorname{Hom}_{L^\infty(\mathbb{G})}(X, L^\infty(\mathbb{G}))$. Since $L^\infty(\mathbb{G})$ is finitely generated as an $L^\infty(\mathbb{G})$-module, Theorem 0.6 in [Lüc98] implies that $\dim_{L^\infty(\mathbb{G})}(J) = \dim_{L^\infty(\mathbb{G})}(\overline{J}^{\text{alg}})$ and thus

$$\beta_0^{(2)}(\mathbb{G}) = 1 - \dim_{L^\infty(\mathbb{G})}(J) = 1 - \dim_{L^\infty(\mathbb{G})}(\bar{J}).$$

Our aim now is to prove that $\bar{J} = L^\infty(\mathbb{G})$. Assume, conversely, that that $\bar{J} \neq L^\infty(\mathbb{G})$ and note that since J is convex, $\bar{J}$ is weak operator closed as well. Because $1 \notin \bar{J}$, the counit $\varepsilon\colon \operatorname{Pol}(\mathbb{G}) \to \mathbb{C}$ extends naturally to the weakly closed subspace

$$\mathbb{C} + \bar{J} = \{\lambda 1 + x \mid \lambda \in \mathbb{C}, x \in \bar{J}\} \subseteq L^\infty(\mathbb{G}),$$

by setting $\varepsilon(\lambda 1 + x) = \lambda$. To see that this extends ε, just note that each element $a \in \operatorname{Pol}(\mathbb{G})$ can be written uniquely as the sum of a scalar and an element from J: $a = \varepsilon(a)1 + (a - \varepsilon(a)1)$. By [KR83] Corollary 1.2.5, the extension $\varepsilon\colon \mathbb{C} + \bar{J} \to \mathbb{C}$ is weakly continuous since its kernel $\bar{J}$ is weakly closed. The Hahn-Banach Theorem therefore allows us to extend ε to a weakly continuous functional, also denoted ε, on all of $B(L^2(\mathbb{G}))$. In particular, ε is weakly continuous on the unit ball of $B(L^2(\mathbb{G}))$ and thus $\varepsilon \in B(L^2(\mathbb{G}))_*$. Denote by η the natural inclusion $\operatorname{Pol}(\mathbb{G}) \subseteq L^2(\mathbb{G})$ and by $W \in B(L^2(\mathbb{G}) \bar{\otimes} L^2(\mathbb{G}))$ the multiplicative unitary for $\mathbb{G}$, which for $x, y \in \operatorname{Pol}(\mathbb{G})$ is given by

$$W^*(\eta(x) \otimes \eta(y)) = (\eta \otimes \eta)(\Delta_{\mathbb{G}}(y)(x \otimes 1)).$$

For any $\omega \in B(L^2(\mathbb{G}))_*$ and any $x \in \mathrm{Pol}(\mathbb{G})$ we have

$$(\omega \otimes \mathrm{id})(W^*)(\eta(x)) = \eta((\omega \otimes \mathrm{id})\Delta_{\mathbb{G}}(x)). \tag{2}$$

This can be proved by a direct calculation when ω has the form $T \mapsto \langle T\eta(a)|\eta(b)\rangle$ and the general case follows from this since $B(L^2(\mathbb{G}))_*$ is the norm closure of the linear span of such functionals [KR86, 7.4.4]. See e.g. Result 1.2.5 in [KV00] for more details. Using the formula (2) with $\omega = \varepsilon$ we therefore obtain

$$(\varepsilon \otimes \mathrm{id})(W^*) = 1.$$

Since ε is weakly continuous, $\varepsilon \otimes \mathrm{id}$ restricts to a $*$-homomorphism

$$\varepsilon \otimes \mathrm{id}\colon L^\infty(\mathbb{G})\bar{\otimes}B(L^2(\mathbb{G})) \longrightarrow B(L^2(\mathbb{G})),$$

and since $W \in L^\infty(\mathbb{G})\bar{\otimes}B(L^2(\mathbb{G}))$ it follows that $(\varepsilon \otimes \mathrm{id})(W) = 1$. This implies that the C^*-algebra of the reduced, C^*-algebraic, dual quantum group $\hat{\mathbb{G}}$ (see (1)) is unital. Therefore $\hat{\mathbb{G}}$ is compact and $\mathbb{G}$ thus both discrete and compact. This forces $C(\mathbb{G})$ to be finite dimensional, contradicting the assumption. □

Remark 2.2. *If $C(\mathbb{G})$ has finite linear dimension N it was proven in* [Kye08b] *Proposition 2.9 that the zeroth L^2-Betti number of $\mathbb{G}$ equals $\frac{1}{N}$. So, by declaring $\frac{1}{\infty} = 0$ we have the formula*

$$\beta_0^{(2)}(\mathbb{G}) = \dim_{\mathbb{C}}(C(\mathbb{G}))^{-1},$$

for any compact quantum group $\mathbb{G}$ of Kac type.

In [CS05] Connes and Shlyakhtenko introduced L^2-homology and L^2-Betti numbers for certain tracial $*$-algebras. For these L^2-Betti numbers we get the following.

Corollary 2.3. *If $\mathbb{G}$ is an infinite, compact quantum group of Kac type then the zeroth Connes-Shlyakhtenko L^2-Betti number of the tracial $*$-algebra* $(\mathrm{Pol}(\mathbb{G}), h_{\mathbb{G}})$ *vanishes.*

Proof. By Theorem 4.1 in [Kye08b] the Connes-Shlyaktenko L^2-Betti numbers of the tracial $*$-algebra $(\mathrm{Pol}(\mathbb{G}), h_{\mathbb{G}})$ are equal, degree by degree, to the L^2-Betti numbers of $\mathbb{G}$. □

3. The zeroth L^2-homology

In this section we will focus on the zeroth L^2-homology module of a compact quantum group $\mathbb{G}$. Since the extended Murray-von Neumann dimension $\dim_{L^\infty(\mathbb{G})}(-)$ is not faithful, it may happen that the homology module $H_0^{(2)}(\mathbb{G})$ is non-trivial although, as we have just seen, its dimension $\dim_{L^\infty(\mathbb{G})}(H_0^{(2)}(\mathbb{G})) = \beta_0^{(2)}(\mathbb{G})$ is zero whenever $\mathbb{G}$ is infinite and of Kac type. In [Lüc02] Lemma 6.36 it is proven, for a discrete group Γ, that the zeroth L^2-homology $H_0^{(2)}(\Gamma)$ is non-vanishing exactly when Γ is an amenable group, and the aim of this section is

to prove an analogue of this result for quantum groups. Since the homology modules $H_n^{(2)}(\mathbb{G}) = \operatorname{Tor}_n^{\operatorname{Pol}(\mathbb{G})}(L^\infty(\mathbb{G}), \mathbb{C})$ are defined also when the Haar state is not a trace[1] we will, throughout this section, allow also compact quantum groups that are not of Kac type. This gives rise to some technical problems since Lück's results on finitely generated Hilbert modules are only available in the tracial setting, but since these technicalities are not of essential nature they are relegated to the appendix. The quantum group analogue of Lück's result takes the following form.

Theorem 3.1. *The zeroth L^2-homology of a compact quantum group $\mathbb{G}$ vanishes if and only if $\mathbb{G}$ is not coamenable.*

Recall that a compact quantum group $\mathbb{G}$ is called coamenable [BMT01] if the counit $\varepsilon\colon \operatorname{Pol}(\mathbb{G}) \to \mathbb{C}$ extends to a character on $C(\mathbb{G})_{\mathrm{red}}$. Lück's proof for discrete groups [Lüc02, 3.36] uses the so-called Kesten condition for amenable groups, and since [Kye08a] Proposition 4.4 (see also [Ban99]) provides us with a Kesten condition for quantum groups we can follow the same strategy here.

Proof. Denote by $(u^\alpha)_{\alpha\in I}$ a complete family of irreducible, unitary corepresentations of $\mathbb{G}$ and denote the dimension (i.e. matrix size) of u^α by $d(u_\alpha)$. Since

$$\operatorname{Pol}(\mathbb{G}) = \operatorname{span}_{\mathbb{C}}\{u_{ij}^\alpha \mid \alpha \in I, 1 \le i,j \le d(u_\alpha)\},$$

we get that

$$\ker(\varepsilon) = \operatorname{span}_{\mathbb{C}}\{u_{ij}^\alpha - \varepsilon(u_{ij}^\alpha)1 \mid \alpha \in I, 1 \le i,j \le d(u_\alpha)\}.$$

Letting T denote the map

$$\bigoplus_{\alpha\in I}\bigoplus_{i,j=1}^{d(u_\alpha)} \operatorname{Pol}(\mathbb{G}) \ni (x_{ij}^\alpha) \longmapsto \sum_{\alpha\in I}\sum_{i,j=1}^{d(u_\alpha)} (u_{ij}^\alpha - \varepsilon(u_{ij}^\alpha)1)x_{ij}^\alpha \in \operatorname{Pol}(\mathbb{G}), \tag{3}$$

we therefore get an exact sequence of right $L^\infty(\mathbb{G})$-modules

$$\bigoplus_{\alpha\in I}\bigoplus_{i,j=1}^{d(u_\alpha)} \operatorname{Pol}(\mathbb{G}) \xrightarrow{T} \operatorname{Pol}(\mathbb{G}) \xrightarrow{\varepsilon} \mathbb{C} \longrightarrow 0.$$

Applying the right exact functor $-\odot_{\operatorname{Pol}(\mathbb{G})} L^\infty(\mathbb{G})$ we obtain another exact sequence

$$\bigoplus_{\alpha\in I}\bigoplus_{i,j=1}^{d(u_\alpha)} L^\infty(\mathbb{G}) \xrightarrow{T} L^\infty(\mathbb{G}) \longrightarrow \mathbb{C} \underset{\operatorname{Pol}(\mathbb{G})}{\odot} L^\infty(\mathbb{G}) \longrightarrow 0.$$

We also denote the induced map by T since it is given by the exact same formula just defined on a bigger domain. Recall that

$$H_0^{(2)}(\mathbb{G}) = \operatorname{Tor}_0^{\operatorname{Pol}(\mathbb{G})}(L^\infty(\mathbb{G}), \mathbb{C}) \simeq L^\infty(\mathbb{G}) \underset{\operatorname{Pol}(\mathbb{G})}{\odot} \mathbb{C},$$

[1] the traciality is only needed in order for their dimension to be defined.

so that $H_0^{(2)}(\mathbb{G}) = 0$ iff T is surjective. We therefore have to prove that T is surjective iff $\mathbb{G}$ is non-coamenable. Assume first that T is surjective; then there exists a finite subset $I_0 \subseteq I$ and a family $\{a_{ij}^\alpha \mid \alpha \in I, 1 \leq i,j \leq d(u_\alpha)\} \subseteq L^\infty(\mathbb{G})$ such that $a_{ij}^\alpha = 0$ when $\alpha \notin I_0$ and

$$1 = T((a_{ij}^\alpha)) = \sum_{\alpha \in I_0} \sum_{i,j=1}^{d(u_\alpha)} (u_{ij}^\alpha - \varepsilon(u_{ij}^\alpha)1)a_{ij}^\alpha.$$

Therefore already the restricted map

$$\bigoplus_{\alpha \in I_0} \bigoplus_{i,j=1}^{d(u_\alpha)} L^\infty(\mathbb{G}) \ni (x_{ij}^\alpha) \overset{T_0}{\longmapsto} \sum_{\alpha \in I_0, i,j} (u_{ij}^\alpha - \varepsilon(u_{ij}^\alpha))x_{ij}^\alpha \in L^\infty(\mathbb{G})$$

is surjective. By Proposition A1 (see the appendix) this implies that also the continuous extension

$$\bigoplus_{\alpha \in I_0} \bigoplus_{i,j=1}^{d(u_\alpha)} L^2(\mathbb{G}) \xrightarrow{\tilde{T}_0} L^2(\mathbb{G})$$

is surjective. By Lemma A3, surjectivity of $\tilde{T}_0$ is equivalent to bijectivity of

$$\tilde{T}_0\tilde{T}_0^* \colon L^2(\mathbb{G}) \to L^2(\mathbb{G}).$$

For notational convenience, we denote by u the direct sum $\oplus_{\alpha \in I_0} u^\alpha$ so that the matrix coefficients defining T_0 are exactly the matrix coefficients of u. A direct calculation shows that $\tilde{T}_0\tilde{T}_0^*$ is the continuous extension of the map $\mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ given by left multiplication with the element

$$2(d(u) - \frac{1}{2}\sum_{i=1}^{d(u)} u_{ii} + u_{ii}^*),$$

proving that

$$\tilde{T}_0\tilde{T}_0^* = 2(d(u) - \frac{1}{2}\sum_{i=1}^{d(u)} \lambda(u_{ii}) + \lambda(u_{ii})^*).$$

So, since $\tilde{T}_0\tilde{T}_0^*$ is bijective zero is not in its spectrum and the unitary corepresentation u therefore has the property that $d(u)$ is not in the spectrum of

$$\frac{1}{2}\sum_{i=1}^{d(u)} \lambda(u_{ii}) + \lambda(u_{ii})^*,$$

which by the Kesten condition for quantum groups [Kye08a, 4.4] implies that $\mathbb{G}$ can not be coamenable.

Assume conversely that $\mathbb{G}$ is coamenable. We then need to prove that the map T in equation (3) is not surjective. The quantum Kesten condition gives that zero is in the spectrum of

$$d(u) - \frac{1}{2}\sum_{i=1}^{d(u)} \lambda(u_{ii}) + \lambda(u_{ii})^*$$

for *any* finite dimensional, unitary corepresentation $u \in \mathbb{M}_{d(u)}(C(\mathbb{G}))$ and by repeating the argument above this means that $\tilde{T}_0\tilde{T}_0^*$ can not be bijective for *any* choice of finite subset $I_0 \subseteq I$. Hence $\tilde{T}_0$ is not surjective for any such I_0. By Proposition A1, this means that T_0 can not be surjective for any choice of finite subset $I_0 \subseteq I$ and hence the original map T can not be surjective either.

$\square$

Remark 3.2. *Theorem 3.1 is a very direct analogue of Lück's result* [Lüc02, 6.36]*, but it also fits well with a result by Connes-Shlyakhtenko* [CS05, 2.6] *stating that the zeroth L^2-homology of a finite factor is non-vanishing exactly when the factor in question is the hyperfinite (a.k.a. amenable!) one.*

Appendix

In this appendix we prove a technical result needed in the proof of Theorem 3.1. Let M be a von Neumann algebra and let φ be a faithful, normal state on M. Consider the GNS-space $H = L^2(M, \varphi)$ with corresponding GNS-representation π and denote the natural inclusion $M \subseteq H$ by η. The result needed is the following.

Proposition A1. *A homomorphism $T\colon M^n \to M^m$ of finitely generated, free, right M-modules is surjective if and only if the continuous extension $\tilde{T}\colon H^n \to H^m$ is surjective.*

Remark A2. *In the case when φ is a tracial state this follows directly from the fact that Lück's L^2-completion functor is exact with exact inverse* [Lüc02, 6.24].

Before giving the proof of Proposition A1 we prove a small result of purely operator theoretic nature. The result is probably well known to operator algebraists, but since we were not able to find a reference we provide the proof for the convenience of the reader.

Lemma A3. *Let H and K be Hilbert spaces and consider an operator $T \in B(H, K)$. Then T is surjective iff TT^* is bijective.*

Proof. If TT^* is bijective we have $K = \mathrm{rg}(TT^*) \subseteq \mathrm{rg}(T)$ and hence T is surjective. If, on the other hand, T is surjective then T^* is injective so if $TT^*\xi = 0$ for some $\xi \in H$ then

$$0 = \langle TT^*\xi \,|\, \xi\rangle = \|T^*\xi\|^2,$$

and $\xi = 0$. Hence TT^* is injective. Consider the restriction $T_0\colon \ker(T)^\perp \to K$ which by assumption is an invertible operator. Therefore also T_0^* is invertible and hence

$$\ker(T)^\perp = \mathrm{rg}\,(T_0^*) = \mathrm{rg}\,(T^*),$$

so that T^* has closed range. Since T is surjective the open mapping theorem [KR83, 1.8.4] gives that T is open and hence that $T(\mathrm{rg}\,(T^*)) = \mathrm{rg}\,(TT^*)$ is a closed subspace of K. Because T is an operator between Hilbert spaces we have

$$\overline{\mathrm{rg}\,(TT^*)} = \overline{\mathrm{rg}\,(T)} = K,$$

and since we just saw that $\mathrm{rg}\,(TT^*)$ is closed we conclude that TT^* is surjective. Thus, TT^* is both injective and surjective and hence an invertible operator. $\square$

Proof of Proposition A1. The proof is divided into three steps.

Step I

We first consider the case $m = n = 1$, in which $T = L_a\colon x \mapsto ax$ for some $a \in M$. Assume first that L_a is surjective. Then there exists $b \in M$ such that $1 = L_a(b) = ab$. Thus

$$\mathrm{id}_H = \pi(1) = \pi(a)\pi(b),$$

and hence $H = \mathrm{rg}\,(\pi(a)\pi(b)) \subseteq \mathrm{rg}\,(\pi(a))$ and $\pi(a)$ is surjective. Assume conversely that $\pi(a)$ is surjective. Then, by Lemma A3, $\pi(a)\pi(a)^* \in \pi(M)$ is invertible and hence there exists $b \in M$ such that

$$\pi(b)\pi(a)\pi(a)^* = \mathrm{id}_H = \pi(a)\pi(a)^*\pi(b).$$

In particular we see

$$\eta(aa^*b) = \pi(aa^*b)\eta(1) = \mathrm{id}_H(\eta(1)) = \eta(1).$$

Because φ is faithful, the map η is an injection and thus $aa^*b = 1$ in M. For arbitrary $c \in M$ we therefore get that $a(a^*bc) = c$ and hence that L_a is surjective.

Step II

Next we consider the case where m equals 1 and n is arbitrary. Since T is assumed (right) M-linear it has the form $(x_1, \dots, x_n) \mapsto \sum_i a_i x_i$ for some elements $a_1, \dots, a_n \in M$. Hence the map $\tilde{T}\colon (\xi_1, \dots, \xi_n) \mapsto \sum_i \pi(a_i)\xi_i$ is the continuous extension of T, and a direct computation shows that the adjoint of $\tilde{T}$ is given by

$$H \ni \xi \longmapsto (\pi(a_1)^*\xi, \dots, \pi(a_n)^*\xi) \in H^n.$$

Assume first that T is surjective. Then there exist $x_1, \dots, x_n \in M$ such that $1 = \sum_i a_i x_i$ and hence

$$\mathrm{id}_H = \sum_i \pi(a_i)\pi(x_i).$$

Thus $\xi = \sum_i \pi(a_i)(\pi(x_i)\xi) = \tilde{T}(\pi(x_1)\xi, \dots, \pi(x_n)\xi)$ for every $\xi \in H$ and $\tilde{T}$ is surjective. Assume conversely that $\tilde{T}$ is surjective and hence, by Lemma A3, that

$\tilde{T}\tilde{T}^*$ is bijective. A direct calculation shows that $\tilde{T}\tilde{T}^* = \pi(c)$ where $c = \sum_i a_i a_i^*$ and by Step I we conclude that $L_c\colon M \to M$ is surjective as well. Thus, for all $b \in M$ there exists $x \in M$ such that

$$b = L_c(x) = (\sum_i a_i a_i^*)x = \sum_i a_i(a_i^* x) = T(a_1^* x, \dots, a_n^* x),$$

and T is therefore surjective.

Step III

Finally we consider the general case. Denote by $q_i\colon M^m \to M$ the projection onto the i-th summand and by $\tilde{q}_i\colon H^m \to H$ its extension. Then T is surjective iff $q_i T$ is surjective for every i and $\tilde{T}$ is surjective iff $\tilde{q}_i\tilde{T} = \widetilde{q_i T}$ is surjective for each i. The statement therefore follows from Step II. $\square$

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